



CONVERGENCE OF PICARD-S HYBRID ITERATION PROCESS FOR STRONGLY ACCRETIVE OPERATOR

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ABSTRACT

Study deals with the convergence of the fixed point of the pseudo-contractive mapping by using Picard-S Hybrid iteration method in the uniformly smooth Banach. This research is a generalization of many of the sources referred to here.

INTRODUCTION

In computational mathematics, an iterative method is a mathematical procedure that generates a sequence of improving approximate solutions for a class of problems. One of this class is to finding the fixed point of operator. There are many theorems talked about finding fixed point for different type of operators by using another iterative, the common theorem in fixed point called Banach fixed point which use Picard iterative. The Banach fixed point theorem have been studied extensively by various authors for approximating fixed points of nonlinear operators in Banach space. C. E. Chidume [3], [5], [7], [8], [11], [13] and [16] are introduced and studied Mann and Ishikawa iteration process to approximate fixed points. Recently, in 1998 Chidume [4],

Liue [15], Osilike [21] and Xu [25] introduced the concepts of Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operators in uniformly smooth Banach spaces. Gloswinski and Le Tallec [9] used three-step iterative schemes to find the approximate solutions of the elasto-viscoplasticity problems, liquid crystal theory, and eigen-value problem. It has been shown in [9] that the three-step iterative scheme gives better numerical results than the two-step and one-step. In 2002, Noor, Rassias and Huang [19] had suggested the three-step iteration process for solving the nonlinear strongly accretive operator equations in real uniformly smooth Banach spaces. Plubteing and Wangkeeree [22] introduced and studied a

multi-step scheme with errors to approximate fixed points of asymptotically nonexpansive operator in a Banach space. In our research we proved the theory of convergence of the fixed point of pseudo-contractive mapping, where the search [10] proved a Picard-S Hybrid is faster iteration among the other as Mann, Ishikawa, Noor, CR iterations and others. As well research [10] used contraction mapping while in our research we used the generalized mapping.

Algorithm 1.1: Let C be a nonempty convex subset of a uniformly smooth Banach space X and let $T: C \rightarrow C$ be operator. For any given $x \in C$, and a fixed positive integer N , the sequence $\{x_n\}$ defined by

$$\begin{aligned} x_0 &\in C \\ x_{n+1} &= Ty_n \\ y_n &= (1 - \alpha_n)Tx_n + \alpha_nTz_n \quad (1.1) \\ z_n &= (1 - \beta_n)x_n + \beta_nTx_n, n \in \mathbb{N} \end{aligned}$$

where $\{\alpha_n\}_{n=0}^\infty$ and $\{\beta_n\}_{n=0}^\infty$ are sequences in $[0,1]$ $\alpha_n + \beta_n = 1$. The above algorithm are called a Picard-S Hybrid iteration.

Our paper is organized as follows. In section two we give two preliminary lemmas and some definitions. In section three we obtain the strong convergence theorems of Picard- S Hybrid iterative scheme for approximating fixed point of nonlinear strongly pseudo-contractive operator or solutions of nonlinear strongly accretive operator.

2- Preliminaries

Now, we recall the well-known concepts and results. Throughout we

assume that X is a real Banach space and X^* is the dual space of X . Let J denoted the normalized duality from X to 2^{X^*} defined by

$$J(x) = \{j \in X^*: \langle x, j \rangle = \|x\| \|j\| = \|x\|^2\} \dots (2.1)$$

Where $\langle \cdot, \cdot \rangle$ denote the generalized duality pairing.

Definition 2.1 [23]: A mapping $T: X \rightarrow X$ is called strongly accretive if there exists a constant $0 < k < 1$ such that for each $x, y \in X$ there is a $j(x - y) \in J(x - y)$ satisfying,

$$\langle Tx - Ty, j(x - y) \rangle \geq k \|x - y\|^2 \dots (2.2)$$

Definition 2.2 [12]: An operator T with domain $D(T)$ and range $R(T)$ in X is called strongly pseudo-contractive if for all $x, y \in D(T)$, there exists $j(x - y) \in J(x - y)$ and constant $0 < k < 1$ such that

$$\langle Tx - Ty, j(x - y) \rangle \leq (1 - k) \|x - y\|^2. \dots (2.3)$$

In [6], can show for any given $f \in X$ the equation $Tx = f$ has a unique solution if $T: X \rightarrow X$ is strongly accretive and continuous, on X is uniformly smooth and $T: X \rightarrow X$ is strongly accretive and demicontinuous (i.e, $x_n \rightarrow x \Rightarrow Tx_n \rightarrow Tx$). Martin [17] has also proved that if $T: X \rightarrow X$ is continuous and accretive then for any given $f \in X$ the equation $x + Tx = f$ has a unique solutions. In the following, we shall give the two lemmas needed in the main results.

Lemma2.3 [25]: Let X be a real uniformly smooth Banach space and let $J: X \rightarrow 2^{X^*}$ be the normalized duality mapping. Then for any $x, y \in X$ we have

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j \rangle,$$

$$\forall j(x + y) \in J(x + y).$$

Lemma 2.4 [15]: Let $\{a_n\}_{n=0}^\infty$, $\{b_n\}_{n=0}^\infty$ and $\{c_n\}_{n=0}^\infty$ three nonnegative real sequences satisfying

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0$$

with $t_n \in [0,1)$,

$$\sum t_n = +\infty, b_n = o(t_n) \text{ and}$$

$$\sum c_n < +\infty. \text{ Then } \lim_{n \rightarrow \infty} a_n = 0.$$

3. RESULTS

In section three, we give prove of the strong convergence of Picard–S Hybrid iterative process defined by (1.1) for the strongly pseudo-contractive operators.

Theorem 3.1: Let X be a real uniformly smooth Banach space and let C be a nonempty bounded closed convex subset of X , and let $T: C \rightarrow C$ be strongly pseudo-contractive mapping with bounded range. Suppose the sequence $\{x_n\}$ be defined by (1.1), where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequence in $[0,1]$ with $\alpha_n + \beta_n = 1$ and satisfying the conditions:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0,$
 $\sum_{n=0}^\infty \beta_n = +\infty;$
- (ii) $\lim_{n \rightarrow \infty} \alpha_n = 0$
 $\sum_{n=0}^\infty \alpha_n < +\infty.$

Then $\{x_n\}$ convergences strongly to the unique common fixed point of T .

Proof: In [12] guarantees the map satisfying condition (2.3) has unique fixed point of q . Since C is bounded and T is bounded rang then $T(TX)$ is bounded, this yields , $\exists M < \infty$ such that

$$\|x_{n+1} - q\| < M, \quad \forall n \in N.$$

$$\begin{aligned} \|z_n - q\|^2 &= \|(1 - \beta_n)x_n + \beta_nTx_n - q\|^2 \\ &= \|(1 - \beta_n)(x_n - q) + \beta_n(Tx_n - q)\|^2 \\ &\leq (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle Tx_n - q, j(z_n - q) \rangle \end{aligned}$$

$$\begin{aligned} &\leq (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n\langle Tx_n - q, j(x_n - q) \rangle \\ &\quad + 2\beta_n\langle Tx_n - q, j(z_n - q) - j(x_n - q) \rangle \\ &\leq (1 - \beta_n)^2\|x_n - q\|^2 + 2\beta_n(1 - k)\|x_n - q\|^2 + 2\beta_n v_n^1 \end{aligned}$$

Where $v_n^1 = \langle Tx_n - q, j(z_n - q) - j(x_n - q) \rangle$. Let $b_n = 2\beta_n v_n^1$, since $\beta_n \rightarrow 0$ as $n \rightarrow \infty$ there exists $m_1 \in N$ such that $\beta_n - k \leq 0$ for all $n \geq m_1$.

Hence

$$\|z_n - q\|^2 \leq (1 - k\beta_n)\|x_n - q\|^2 + 2\beta_n v_n^1 \quad n \geq m_1.$$

Now,

$$\begin{aligned} \|y_n - q\|^2 &= \|(1 - \alpha_n)Tx_n + \alpha_n Tz_n - q\|^2 \\ &= \|(1 - \alpha_n)(Tx_n - q) + \alpha_n(Tz_n - q)\|^2 \\ &\leq (1 - \alpha_n)^2\|Tx_n - q\|^2 + 2\alpha_n \langle Tz_n - q, j(y_n - q) \rangle \\ &\leq (1 - \alpha_n)^2(1 - k)\|x_n - q\|^2 + 2\alpha_n(1 - k)\|z_n - q\|^2 \\ &\quad + 2\alpha_n \langle Tz_n - q, (y_n - q) - j(z_n - q) \rangle \\ &= (1 - \alpha_n)^2(1 - k)\|x_n - q\|^2 + 2\alpha_n(1 - k)\|z_n - q\|^2 + 2\alpha_n v_n^2 \end{aligned}$$

Where

$$v_n^2 = \langle Tz_n - q, j(y_n - q) - j(z_n - q) \rangle$$

This implies that

$$\begin{aligned} \|y_n - q\|^2 &\leq 2(1 - k)\|x_n - q\|^2 + 4(1 - k)\alpha_n \beta_n v_n^1 + 2\alpha_n v_n^2 \\ &\leq (1 - k\alpha_n \beta_n)\|x_n - q\|^2 + 4(1 - k)\alpha_n \beta_n v_n^1 + 2\alpha_n v_n^2 \end{aligned}$$

Next, we explain

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|Ty_n - q\|^2 \\ &= \langle Ty_n - q, j(y_n - q) \rangle \\ &\leq (1 - k)\|y_n - q\|^2 \\ &\leq 2(1 - k)^2(1 - k \alpha_n \beta_n)\|x_n - q\|^2 + \\ &\quad b_n + c_n \\ &\leq (1 - k \alpha_n \beta_n)\|x_n - q\|^2 + b_n + c_n \end{aligned}$$

Where $b_n = 4(1 - k)^2 \alpha_n \beta_n v_n^1$, $c_n = 2(1 - k) \alpha_n v_n^2$.

This implies that

$$\|x_{n+1} - q\|^2 \leq a_n \|x_n - q\|^2 + b_n + c_n \tag{3.4}$$

where $a_n = (1 - k\beta_n)$

Then (3.4) reduces to

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, n \geq m_1$$

Where $\|x_n - q\|^2 = a_n$ and $k\beta_n = t_n$.

Now, we show that

$\langle Tx_n - q, j(z_n - q) - j(x_n - q) \rangle \rightarrow 0$ as $n \rightarrow \infty$,
and $\langle Tz_n - q, j(z_n - q) - j(y_n - q) \rangle \rightarrow 0$ as $n \rightarrow \infty$. Since $\lim_{n \rightarrow \infty} \beta_n = 0$,

$$\begin{aligned} \|(z_n - q) - (x_n - q)\| &= \|(1 - \beta_n)x_n + \beta_n Tx_n - x_n\| \\ &= \|x_n - \beta_n Tx_n + \beta_n Tx_n - x_n\| \\ &= \beta_n \|x_n - Tx_n\| \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$.

$$\begin{aligned} \|(z_n - q) - (y_n - q)\| &= \|(1 - \beta_n)x_n + \beta_n Tx_n \\ &\quad - \beta_n Tx_n - (1 - \beta_n)Tz_n\| \\ &= \|(1 - \beta_n)(x_n - Tz_n)\| \\ &= (1 - \beta_n)\|x_n - Tz_n\| \rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This, we give $\{x_n - q\}_{n=1}^\infty$, $\{z_n - q\}_{n=1}^\infty$ and $\{y_n - q\}_{n=1}^\infty$ are bounded sets, from Lemma 2.2, since X is uniformly smooth Banach space, j is single valued and

uniformly continuous on any bonded subsets of X this implies that

$$\langle Tx_n - q, j(z_n - q) - j(x_n - q) \rangle \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and

$$\langle Tz_n - q, j(z_n - q) - j(y_n - q) \rangle \rightarrow 0 \text{ as } n \rightarrow \infty$$

This implies that

$$\lim_{n \rightarrow \infty} \left(\frac{b_n}{k\beta_n}\right) = 0. \text{ Therefore } b_n = o(k\beta_n).$$

We note that $\sum_{n=1}^\infty c_n < \infty$. Then, by lemma 2.3, $\lim_{n \rightarrow \infty} \|x_n - q\| = 0$ and therefore $x_n \rightarrow q \in X$ as $n \rightarrow \infty$. Recently, we will show that q is a unique common fixed point of T . Let $q^* \in F$ is a set of fixed point of T , we can show

$$\begin{aligned} \|q - q^*\| &= \langle Tq - Tq^*, j(q - q^*) \rangle \\ &\leq (1 - k)\|q - q^*\|, \end{aligned}$$

since $k \in (0,1)$, we have the equality that $q = q^*$. \square

Corollary 3.2: Let T be maps from a real uniformly smooth Banach space X to itself. Define $S: X \rightarrow X$ by $Sx = x - Tx + f$, where $f \in X$ is fixed and suppose that the range of S is bounded. Then the sequence $\{x_n\}$ generated by Picard-S Hybrid iteration is defined by

$$\begin{aligned} x_0 &\in X \\ x_{n+1} &= Sy_n \\ y_n &= (1 - \alpha_n)Sx_n + \alpha_n Sz_n \\ z_n &= (1 - \beta_n)x_n + \beta_n Sx_n, n \in \mathbb{N} \end{aligned}$$

Satisfy the following property:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$,
- (ii) $\sum_{n=0}^\infty \beta_n = +\infty$;
- (iii) $\lim_{n \rightarrow \infty} \alpha_n = 0$
- (iv) $\sum_{n=0}^\infty \alpha_n < +\infty$.

And $\alpha_n + \beta_n = 1$ for all $n \in \mathbb{N}$. Then the sequence $\{x_n\}$ convergence strongly to the unique solution of the equation $Tx = f$.

Proof: Let T satisfy the condition (2.2),

Then

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2 \quad \text{for all } x, y \in X$$

Now, we show that

$$\begin{aligned} \langle Sx - Sy, j(x - y) \rangle &= \langle (x - Tx + f) - (y - Ty + f), j(x - y) \rangle \\ &= \langle x - y, j(x - y) \rangle - \langle Tx - Ty, j(x - y) \rangle \\ &\leq \|x - y\|^2 - k\|x - y\|^2 = (1 - k)\|x - y\|^2 \end{aligned}$$

So, S is strongly pseudo-contractive mapping. Since the range of S is bounded.

The application of the same as the previous steps in theory (3.1) we note that the sequence $\{x_n\}$ convergence strongly to the solution of the equation $Tx = f$.

Corollary 3.3:([19]) Let X be a real uniformly smooth Banach space and let C be a nonempty bounded closed convex subset of X, and let T: C → C be a strongly pseudo-contractive mapping. Let q be a fixed point of T and let the Ishikawa iteration sequence $\{x_n\}$ be defined by (1.2): satisfying:

$$\lim_{n \rightarrow \infty} \alpha_n = 0, \quad \lim_{n \rightarrow \infty} \beta_n = 0, \quad \lim_{n \rightarrow \infty} \gamma_n = 0 \quad \sum_{n=0}^{\infty} \alpha_n = +\infty.$$

Then $\{x_n\}$ converges strongly to the unique fixed point of T.

For $N = 2$, $T_1 = T_2$, $\alpha_n = \alpha_n^1$, $\beta_n = \beta_n^1$, $\gamma_n = \gamma_n^1$, $\alpha_n = \alpha_n^1$, $\beta_n = \beta_n^1$, $\gamma_n = \gamma_n^1$, in theorem 3.1, we can

obtain Ishikawa-type Convergence result for two mappings.

Corollary 3.4:([25]) Let X be a real uniformly smooth Banach space and let C be a nonempty bounded closed convex subset of X, and let T: C → C be a strongly pseudo-contractive mapping. Let q be a fixed point of T and let the Ishikawa iteration sequence $\{x_n\}$ be defined by (1.3): satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} \beta'_n = 0$;
- (iii) $\lim_{n \rightarrow \infty} \gamma'_n = 0$, $\lim_{n \rightarrow \infty} \gamma_n = 0$, and $\sum_{n=0}^{\infty} \gamma_n < +\infty$.

Then $\{x_n\}$ converges strongly to the unique fixed point of T.

Now, we give the multi-step iterative process with errors for solving nonlinear strongly accretive operators $T_i = f$ for $i=1, \dots, N$. Where it was based on theory 3.2 and theory 3.3 in the proof of the following theory.

Theorem 3.5:([12]) Let X be a real uniformly smooth Banach space and let $T_1, \dots, T_N: X \rightarrow X$ be a strongly accretive mappings. For a fixed $f \in X$, define $S_1, \dots, S_N: X \rightarrow X$ by $S_i x = x - T_i x + f$ for all $i=1, \dots, N$ and suppose that each range of S_i are bounded. For arbitrary $x_1 \in X$ the sequence $\{x_n\}$ with errors is defined by

$$\begin{cases} x_1 \in X, \\ x_n^1 = \alpha_n^1 x_n + \beta_n^1 S_1 x_n + \gamma_n^1 u_n^1, \\ x_n^2 = \alpha_n^2 x_n + \beta_n^2 S_2 x_n^1 + \gamma_n^2 u_n^2, \\ \vdots \\ x_{n+1} = x_n^N = \alpha_n^N x_n + \beta_n^N S_N x_n^{N-1} + \gamma_n^N u_n^N, \\ n \geq 1 \end{cases}$$

where $\{u_n^1\}, \{u_n^2\}, \dots, \{u_n^N\}$, are bounded sequences in X and $\{\alpha_n^1\}, \{\alpha_n^2\}, \dots, \{\alpha_n^N\}, \{\beta_n^1\}, \{\beta_n^2\}, \dots, \{\beta_n^N\}, \{\gamma_n^1\}, \{\gamma_n^2\}, \dots, \{\gamma_n^N\}$ are sequences in $[0,1]$ with $\alpha_n^i + \beta_n^i + \gamma_n^i = 1$ for $i=1, \dots, N$ and satisfying the following conditions:

- (i) $\lim_{n \rightarrow \infty} \beta_n^N = 0$ for all $i=1, 2, \dots, N$ and $\sum_{n=0}^{\infty} \beta_n^N = +\infty$;
- (ii) $\lim_{n \rightarrow \infty} \gamma_n^i = 0$, for all $i=1, 2, \dots, N$ and $\sum_{n=0}^{\infty} \gamma_n^N < +\infty$.

If the systems of operator $T_1x = f, \dots, T_Nx = f$ has solution in X , then the sequence $\{x_n\}$ converges strongly to the unique solution of operator equations $T_1x = f, \dots, T_Nx = f$.

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